J-18

ON THE CONSISTENCY OF TWO CONVENTIONAL COUPLED-MODE FORMULATIONS FOR PARALLEL DIELECTRIC WAVEGUIDES

Hung-chun Chang

Department of Electrical Engineering
National Taiwan University
Taipei, Taiwan, Republic of China

ABSTRACT

Two different formulations of the conventional coupled-mode theory, one based on the partition of the total field and the other based on the projections of the same field on local complete sets, are compared. They are proved to be consistent with each other when the individual waveguide mode overlap integrals are appropriately taken into account.

INTRODUCTION

The use of coupled-mode theory for analyzing the coupling between parallel dielectric waveguides has been well known in the fields of integrated optics and fiber optics [1]-[3]. Recently, much interest has been shown in obtaining "improved" coupled-mode equations for dielectric guides [4]-[6], in which the integrated overlap of the individual waveguide modes over all space, that was usually ignored in the "conventional" theories, is included. It is claimed that these new equations could extend the coupled-mode theory to more strongly coupled waveguide geometries and it has been demonstrated that the accuracy is significantly increased for coupling between nonidentical slab waveguides. For the case of coupling between circular cylindrical guides, it has recently been pointed out that the simpler conventional theory occurs to give better results for coupling between touching identical optical fiber cores [7].

The objective of this paper is to compare two different forms of the conventional coupled-mode theory, one based on the partition modal amplitudes [1], [3] and the other based on the projection modal amplitudes of the total field [2]. The partition modal amplitude and the projection modal amplitude will be defined in the next section and their meanings will become clear there. We show the consistency of these two formulations through the derivation of the "improved" equations and the relationship between the two different modal amplitudes. It is found that proper inclusion of the individual-guide overlap integral in the initial conditions, or the consideration of butt coupling [5], is essential in establishing this consistency. For simplicity, we discuss the theory for the coupling between two parallel dielectric waveguides, each supports only one guided mode when in isolation.

COUPLED-MODE EQUATIONS CONTAINING OVERLAP INTEGRALS

Consider two parallel waveguides denoted "a" and "b", which may differ in shape, size, and/or index of refraction. Let $\epsilon = \epsilon\,(x,y)$ be the dielectric constant of the coupled system and $\epsilon^{(a)} = \epsilon^{(a)}\,(x,y)$ and $\epsilon^{(b)} = \epsilon^{(b)}\,(x,y)$ be the dielectric constants of the individual waveguides singly embedded in the surrounding medium. The definitions lead to the understanding that $\epsilon - \epsilon^{(a)}$ and $\epsilon - \epsilon^{(b)}$ are the perturbations to the respective waveguide dielectric distribution. We assume $\exp(-i\omega t)$ field variations so that propagation along the guides is described by $\exp(i\beta z)$. Let the fields $\bar{E}\,(x,y,z) = \bar{E}_t\,(x,y,z) + \bar{E}_z\,(x,y,z)$ and $\bar{H}\,(x,y,z) = \bar{H}_t(x,y,z) + \bar{H}_z(x,y,z)$ be the unknown solutions which satisfy Maxwell's equations plus the boundary conditions of the coupled system with $\epsilon\,(x,y)$, where the subscripts t and z correspond to the transverse and the longitudinal components, respectively. The modal vectorial fields of waveguide "a" are of the form

$$\bar{E}_p^{(a)}\,(x,y,z) = \bar{e}_p^{(a)}\,(x,y)\,e^{i\beta_p^{(a)}z} = [\,\bar{e}_{pt}^{(a)} + \bar{e}_{pz}^{(a)}\,]\,e^{i\beta_p^{(a)}z} \tag{1a}$$

$$\bar{H}_p^{(a)}\,(x,y,z) = \bar{h}_p^{(a)}(x,y)\,e^{i\beta_p^{(a)}z} = [\,\bar{h}_{pt}^{(a)} + \bar{h}_{pz}^{(a)}\,]\,e^{i\beta_p^{(a)}z} \tag{1b}$$

where p = 1 corresponds to the guided mode popagating in the +z direction and $|\,p\,| > 1$ correspond to the radiation modes. We have the following relationships between the forward and backward propagating modes for the transverse and longitudinal field components and the modal propagation constant,

$$\bar{e}_{(-p)t}^{(a)}(x,y) = \bar{e}_{pt}^{(a)}(x,y), \quad \bar{e}_{(-p)z}^{(a)}(z,y) = -\bar{e}_{pz}^{(a)}(x,y) \tag{2a}$$

$$\bar{h}_{(-p)t}^{(a)}(x,y) = -\bar{h}_{pt}^{(a)}(x,y), \bar{h}_{(-p)z}^{(a)}(x,y) = \bar{h}_{pz}^{(a)}(x,y) \tag{2b}$$

$$\beta_{-p}^{(a)} = -\beta_p^{(a)} \tag{2c}$$

and the orthogonality relation

$$\iint_{-\infty}^{\infty} \hat{z}\cdot(\bar{e}_{pt}^{(a)} \times \bar{h}_{qt}^{(a)})\,d\,x\,dy = 0, \text{ for } \beta_p^{(a)} \neq \pm\beta_q^{(a)}. \tag{3}$$

Similar expressions can be written for waveguide "b" when (a) in (1)–(3) is replaced by (b).

Because of the completeness of the transverse-modal fields of the individual waveguides, we have the modal expansions for the unknown fields

$$\bar{E}_t(x,y,z) = \sum_p a_p(z)\,\bar{e}^{(a)}_{pt}(x,y) = \sum_p b_p(z)\,\bar{e}^{(b)}_{pt}(x,y) \quad (4a)$$

and $$\bar{H}_t(x,y,z) = \sum_p a_p(z)\,\bar{h}^{(a)}_{pt}(x,y) = \sum_p b_p(z)\,\bar{h}^{(b)}_{pt}(x,y). \quad (4b)$$

We call $a_p(z)$ and $b_p(z)$ the projection modal amplitudes of the unknown fields. By following the derivation given in [2] but using the unconjugated reciprocity theorem, it can be shown [8] that $a_1(z)$ and $b_1(z)$ obey the following differential equations

$$\frac{d\,a_1(z)}{dz} = i\beta_1^{(a)} a_1(z) + i\,\frac{\omega}{4}\iint_{-\infty}^{\infty}(\epsilon - \epsilon^{(a)})\,[\bar{e}^{(a)}_{1t} - \bar{e}^{(a)}_{1z}]\cdot[\bar{E}_t + \bar{E}_z]\,dx\,dy \quad (5a)$$

$$\frac{d\,b_1(z)}{dz} = i\beta_1^{(b)} b_1(z) + i\,\frac{\omega}{4}\iint_{-\infty}^{\infty}(\epsilon - \epsilon^{(b)})\,[\bar{e}^{(b)}_{1t} - \bar{e}^{(b)}_{1z}]\cdot[\bar{E}_t + \bar{E}_z]\,dxdy \quad (5b)$$

It should be noted that (5a) and (5b) are exact, since no approximations have been used in the derivation. At this point, we define

$$C_{uv} = \frac{1}{2}\iint_{-\infty}^{\infty}\hat{z}\cdot(\bar{e}^{(v)}_{1t}\times\bar{e}^{(u)}_{1t})\,dxdy,\; u,v = a,b. \quad (6)$$

Note that C_{ab} and C_{ba} are the overlap integrals which describe the individual waveguide mode overlap. In deriving (5a) and (5b), we have assumed $C_{aa} = C_{bb} = 1$.

Now, if we assume that $\bar{E}_t(x,y,z)$ and $\bar{H}_t(x,y,z)$ can be expressed as

$$\bar{E}_t(x,y,z) = A(z)\,\bar{e}^{(a)}_{1t}(x,y) + B(z)\,\bar{e}^{(b)}_{1t}(x,y) \quad (7a)$$

and

$$\bar{H}_t(x,y,z) = A(z)\,\bar{h}^{(a)}_{1t}(x,y) + B(z)\,\bar{h}^{(b)}_{1t}(x,y), \quad (7a)$$

then from Maxwell's equations we have [2], [6]

$$\bar{E}_z(x,y,z) = A(z)\,\frac{\epsilon^{(a)}}{\epsilon}\,\bar{e}^{(a)}_{1z}(x,y) + B(z)\,\frac{\epsilon^{(b)}}{\epsilon}\,\bar{e}^{(b)}_{1z}(x,y) \quad (8a)$$

and

$$\bar{H}_z(x,y,z) = A(z)\,\bar{h}^{(a)}_{1z}(x,y) + B(z)\,\bar{h}^{(b)}_{1z}(x,y). \quad (8b)$$

Consequently, (5a) and (5b) can be reduced to the following form

$$\frac{d\,a_1(z)}{dz} = i\beta_1^{(a)} a_1(z) + i\,\tilde{K}_{aa}\,A(z) + i\,\tilde{K}_{ab}\,B(z) \quad (9a)$$

$$\frac{d\,b_1(z)}{dz} = i\beta_1^{(b)} b_1(z) + i\,\tilde{K}_{bb}\,B(z) + i\,\tilde{K}_{ba}\,A(z) \quad (9b)$$

where

$$\tilde{K}_{uv} = \frac{\omega}{4}\iint_{-\infty}^{\infty}(\epsilon - \epsilon^{(u)})\,[\bar{e}^{(u)}_{1t}\cdot\bar{e}^{(v)}_{1t} - \frac{\epsilon^{(v)}}{\epsilon}\,\bar{e}^{(u)}_{1z}\cdot\bar{e}^{(v)}_{1z}]\,dxdy \qquad u,v = a,b. \quad (10)$$

We call $A(z)$ and $B(z)$ the partition modal amplitudes, since the total field is assumed to be partitioned into two parts with amplitudes $A(z)$ and $B(z)$, respectively.

If we choose $A(z)$ and $B(z)$ such that

$$a_1(z) = A(z) + C_{ab}\,B(z) \quad (11a)$$

$$b_1(z) = B(z) + C_{ba}\,A(z) \quad (11b)$$

and substitute into (9a) and (9b), it can be shown [8] that the resulting coupled equations for $A(z)$ and $B(z)$ are identical to those derived in [4] with $A(z) = U(z)$ and $B(z) = V(z)$. The conditions (11a) and (11b) are equivalent to requiring the residual fields be orthogonal to the two individual guided modes [4]. Note that (11a) and (11b) are obtained by the cross product of $h^{(a)}_{1t}$ and $h^{(b)}_{1t}$ with (7a). If we cross product $e^{(a)}_{1t}$ and $e^{(b)}_{1t}$ with (7b), the C_{ab} and C_{ba} should be exchanged. The overlap integrals C_{ab} and C_{ba} are slightly different due to the possible small difference in the wave impedance of the two guided modes. Due to this subtlety, we might as well choose to write

$$a_1(z) = A(z) + \bar{C}\,B(z) \quad (12a)$$

$$b_1(z) = B(z) + \bar{C}\,A(z) \quad (12b)$$

where $C = (C_{ab} + C_{ba})/2$. Substituting (12a) and (12b) into (9a) and (9b) and solve them simultaneously, we obtain the coupled-mode equations, which are the same as those given in [6],

$$\frac{d\,A(z)}{dz} = i\,\gamma_a\,A(z) + i\,K_{ab}\,B(z) \quad (13a)$$

$$\frac{d\,B(z)}{dz} = i\,\gamma_b\,B(z) + i\,K_{ba}\,A(z) \quad (13b)$$

where

$$\gamma_a = \beta_1^{(a)} + [\,\tilde{K}_{aa} - \tilde{K}_{ab}\,\bar{C}\,]/(1-\bar{C}^2) \quad (14a)$$

$$\gamma_b = \beta_1^{(b)} + [\,\tilde{K}_{bb} - \tilde{K}_{ba}\,\bar{C}\,]/(1-\bar{C}^2) \quad (14b)$$

$$K_{ab} = [\,\tilde{K}_{ba} - \tilde{K}_{bb}\,\bar{C}\,]/(1-\bar{C}^2) \quad (14c)$$

$$K_{ba} = [\,\tilde{K}_{ab} - \tilde{K}_{aa}\,\bar{C}\,]/(1-\bar{C}^2) \quad (14d)$$

In writing (14a)-(14d), the relation

$$\tilde{K}_{ba} - \tilde{K}_{ab} = \bar{C}\,(\beta_1^{(a)} - \beta_1^{(b)}) \quad (15)$$

has been used. Note that this relation can be derived from the reciprocity theorem and is an exact one [6].

THE CONVENTIONAL COUPLED-MODE EQUATIONS

The equations (13a) and (13b) contain the overlap integrals C_{ab} and C_{ba}. In the conventional coupled-mode theory, these overlap integrals are assumed small and ignored. In doing so, (13a) and (13b) become

$$\frac{d\,A(z)}{dz} = i\,(\beta_1^{(a)} + \widetilde{K}_{aa})\,A(z) + i\,\widetilde{K}_{ba}\,B(z) \tag{16a}$$

$$\frac{d\,B(z)}{dz} = i\,(\beta_1^{(b)} + \widetilde{K}_{bb})\,B(z) + i\,\widetilde{K}_{ab}\,A(z) \tag{16b}$$

which are the equations given in [3]. If $\widetilde{K}_{aa}$ and $\widetilde{K}_{bb}$, which are one-order smaller than $\widetilde{K}_{ba}$ and $\widetilde{K}_{ab}$, are also ignored, (16a) and (16b) would lead to the equations derived in [1]. We call (16a) and (16b) the conventional coupled-mode equations based on the partition modal amplitudes.

Now consider (9a), (9b) and (12a), (12b). We solve (12a) and (12b) for A(z) and B(z) and obtain

$$A(z) = [\,a_1(z) - \overline{C}\,b_1(z)\,] \,/\, (1 - \overline{C}^2). \tag{17a}$$
$$B(z) = [\,b_1(z) - \overline{C}\,a_1(z)\,] \,/\, (1 - \overline{C}^2). \tag{17b}$$

Substituting (17a) the (17b) into (9a) and (9b) and ignoring the similar terms as we derived (16a) and (16b), we obtain

$$\frac{d\,a_1(z)}{dz} = i\,(\beta_1^{(a)} + \widetilde{K}_{aa})\,a_1(z) + i\,\widetilde{K}_{ab}\,b_1(z) \tag{18a}$$

$$\frac{d\,b_1(z)}{dz} = i\,(\beta_1^{(b)} + \widetilde{K}_{bb})\,b_1(z) + i\,\widetilde{K}_{ba}\,a_1(z) \tag{18b}$$

which we call the conventional coupled-mode equations based on the projection modal amplitudes.

Comparing (16a), (16b) and (18a), (18b), we observe that the parameters $\widetilde{K}_{ab}$ and $\widetilde{K}_{ba}$ appear in different places. However, if $\widetilde{K}_{aa}$ and $\widetilde{K}_{bb}$ are ignored, it is easy to show that these two sets of equations are equivalent through the relation (15) and equations (12a) and (12b). The essential point, due to (12a) and (12b), is that if A(0) and B(0) are the initial conditions for (16a) and (16b), then $a_1(0) = A(0) + \overline{C}B(0)$ and $b_1(0) = B(0) + \overline{C}A(0)$ should be used as the initial conditions for (18a) and (18b). Using such initial conditions, it can be shown that the solutions to (16) and (18) satisfy the relations

$$a_1(z) = A(z) + \overline{C}\,B(z)$$
$$= \Big\{ a_1(0)\,[\,\cos(\psi z) - i\,\frac{\beta_1^{(b)} - \beta_1^{(a)}}{2\psi}\,\sin(\psi z)] + b_1(0)\,\frac{i\,\widetilde{K}_{ab}}{\psi}\,\sin(\psi z) \Big\}\, e^{\,i\frac{\beta_1^{(a)} + \beta_1^{(b)}}{2} z} \tag{19a}$$

$$b_1(z) = B(z) + \overline{C}\,A(z)$$
$$= \Big\{ a_1(0)\,\frac{i\,\widetilde{K}_{ba}}{\psi}\,\sin(\psi z) + b_1(0)\,[\,\cos(\psi z) + i\,\frac{\beta_1^{(b)} - \beta_1^{(a)}}{2\psi}\,\sin(\psi z)] \Big\}\, e^{\,i\frac{\beta_1^{(a)} + \beta_1^{(b)}}{2} z} \tag{19b}$$

where $$\psi = \sqrt{[\,\beta_1^{(b)} - \beta_1^{(a)}\,]^2/4 + \widetilde{K}_{ab}\,\widetilde{K}_{ba}}, \tag{20}$$

and we have ignored $\widetilde{K}_{aa}$ and $\widetilde{K}_{bb}$.

In the study of power coupling between two optical waveguides [4], [5], the power output or the power remaining in individual guide is estimated as $|A(z) + \overline{C}\,B(z)|^2$ or $|B(z) + \overline{C}\,A(z)|^2$, which is simply $|a_1(z)|^2$ or $|b_1(z)|^2$. Therefore, we think it is more advantageous to using the coupled-mode equations based on projection modal amplitudes such as (18). Making use of the concept of "butt coupling" [5], it is easy to understand that initial conditions such as A(0) = 1, B(0) = 0 for the equations based on partition modal amplitudes lead to a nonzero initial value for $b_1(z)$.

CONCLUSION

Two different formulations of the conventional coupled-mode theory for dielectric waveguides, one based on the partition modal amplitudes and the other based on the projection modal amplitudes, have been examined. We have shown the consistency of these two formulations by using a relation among the coupling coefficients, the overlap integral, and the propagation constants (equation (15)) and considering the relationship between the partition modal amplitudes and the projection modal amplitudes (equation (12)). The individual waveguide mode overlap integral, or the concept of butt coupling, has been found to play an essential role in reaching the conclusion.

ACKNOWLEDGEMENTS

This work was supported by the National Science Council of the Republic of China under grant NSC76-0608-E002-05. The author would like to thank Prof. S.L. Chuang of University of Illinois for kindly providing the preprint of reference [6].

REFERENCES

[1] D. Marcuse, Light Transmission Optics. New York: Van Nostrant Reinhold, 1972.

[2] A. W. Snyder, "Coupled-mode theory for optical fibers," J. Opt. Soc. Amer., vol. 62, pp. 1267-1277, 1972.

[3] H. F. Taylor and A. Yariv, "Guided wave optics," Proc. IEEE, vol. 62, pp. 1044-1060, 1974.

[4] A. Hardy and W. Streifer, "Coupled mode theory of parallel waveguides," J. Lightwave Technol., vol. LT-3, pp. 1135-1146, 1985.

[5] E. Marcatili, "Improved coupled-mode equations for dielectric guides," IEEE J. Quantum Electron., vol. QE-22, pp. 988-993, 1986.

[6] S. L. Chuang, "A coupled-mode formulation by reciprocity and a variational principle," J.Lightwave Technol., vol. LT-5, pp. 5-15, 1987.

[7] A. Ankiewicz, A. W. Snyder, and X. H. Zheng, "Coupling between parallel optical fiber cores-critical examination," J. Lightwave Technol., vol. LT-4, pp. 1317-1323, 1986.

[8] H. C. Chang, "Coupled-mode equations for dielectric waveguides based on projection modal amplitudes and on partition modal amplitudes," to be published.